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Evaluation of the Coulomb force via the Fredholm integral equation

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Abstract. The electrostatic force between two conducting spheres is solved exactly from Coulomb's law. The force is evaluated in two steps: (i) the charge distribution on each of the spheres is calculated as a function of separation distance by way of a solution to the Fredholm integral equation; (ii) the effect of the known charge distribution is integrated to obtain the electrostatic force. Evaluation of the electrostatic force is fast because the series expression for the charge distribution is 'super converging', a characteristic trait for some Fredholm-type equations.

1. Introduction

Electrostatic forces are readily estimated by the simple dipole expression when particle spacing is relatively large and particles are assumed to be small. To improve accuracy the simple dipole expression can be replaced by the more general multipolar expression [1]. A series expression for a finite number of multipoles is often adequate for many practical applications if the interaction is either monotonically attractive or monotonically repulsive. Alternatively, the method of images yields the electrostatic force directly using the classical series expression [2] which is also written in a modified form to speed up convergence [3], or by using approximate expressions [4].

In this contribution we present the closed form series solution of the Coulomb force [5] obtained from Gauss' [6] generalization of the electrostatic force for distributed charges. The solution for the charge distribution is obtained directly by way of the Fredholm integral equation of the second kind [7] whose solution is known to be rapidly convergent. The method presented is direct and systematic and can be readily generalized to many-body systems.

1.1. Coulomb's law

Coulomb's law for point charges is readily generalized to account for the electrostatic force due to an ensemble of charges residing on two macroscopic surfaces. We consider the electrostatic force between two conducting spheres of radius a_1 and a_2 connected to an external power supply where the top sphere is raised to a constant potential of V_1 volts (in steady state), acquiring a surface charge density σ_1 and the bottom sphere is raised to a constant potential of V_2 volts (in steady state), acquiring a surface charge density σ_2. The Coulomb force [5] on the top sphere is then solely due to the charged bottom sphere, and is given by

$$\vec{F} = K \int \mathrm{d}Q_1(\vec{X}_1)(-\vec{\nabla}_{\vec{X}_1}) \int \frac{\mathrm{d}Q_2(\vec{x}_2)}{|\vec{X}_1 - \vec{x}_2|} \tag{1}$$

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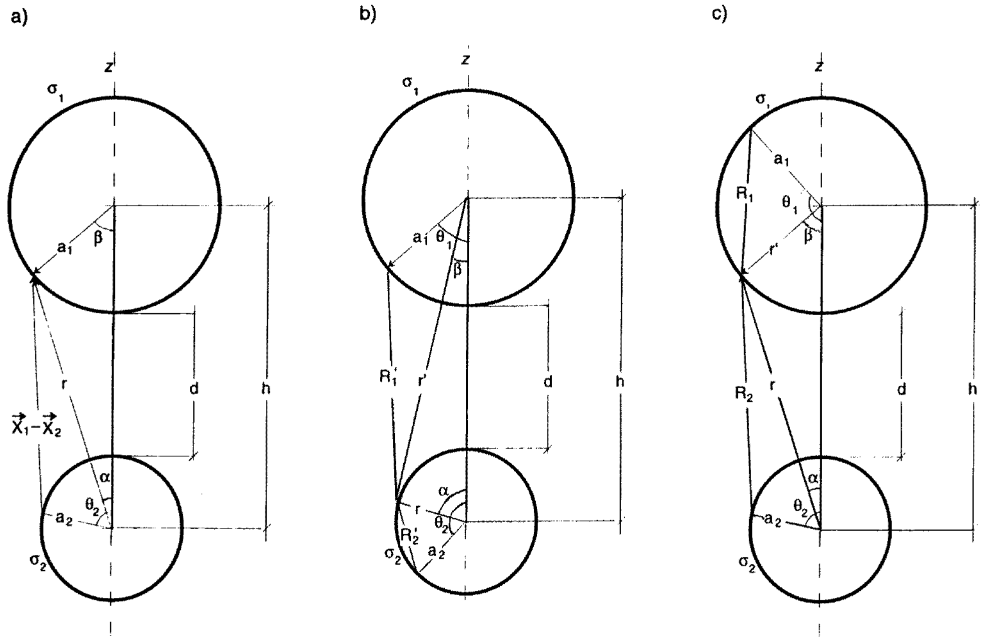

Figure 1. Schematic representation of two spheres each having a uniform distribution of potential. Potentials, charge densities, and the radii on the top and bottom spheres are denoted V_1, σ_1, a_1 and V_2, σ_2, a_2, respectively.

where $\vec{X}_1 \equiv (X_1, Y_1, Z_1)$ and $\vec{x}_2 \equiv (x_2, y_2, z_3)$ are points on the top and bottom spheres, respectively. For a spherical coordinate system equation (1) is written

$$\vec{F} = K \int_0^{\pi} \int_0^{2\pi} a_1^2 \sin\beta \, \mathrm{d}\beta \, \mathrm{d}\phi' \sigma_1(\beta) (-\vec{\nabla}_{[r,\alpha,\phi]}) \int_0^{\pi} \int_0^{2\pi} a_2^2 \sin\theta_2 \, \mathrm{d}\theta_2 \, \mathrm{d}\phi_2 \times \frac{\sigma_2(\theta_2)}{\sqrt{r^2 + a_2^2 - 2a_2 r[\cos\alpha\cos\theta_2 + \sin\alpha\sin\theta_2\cos(\phi - \phi_2)]}} \tag{2}$$

where (figure 1(*a*)) $r = \sqrt{a_1^2 + h^2 - 2a_1 h \cos\beta}$ and $\cos\alpha = \frac{h - a_1\cos\beta}{\sqrt{a_1^2 + h^2 - 2a_1 h\cos\beta}}$. Note the restriction that Coulomb's law is not valid upon contact ($d = h - a_1 - a_2 = 0$), whereby $|\vec{X}_1 - \vec{x}_2| \neq 0$.

Equation (2) can also be written in terms of Legendre polynomials, $P_\ell(x)$:

$$\vec{F} = \hat{z} K (2\pi)^2 (a_1 a_2)^2 \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(\ell + m + 1)!}{\ell! m!} \frac{a_2^\ell a_1^m}{h^{\ell+m+2}} \int_0^{\pi} \sin\beta \, \mathrm{d}\beta \sigma_1(\beta) P_m(\cos\beta) \int_0^{\pi} \sin\theta_2 \, \mathrm{d}\theta_2 \sigma_2(\theta_2) P_\ell(\cos\theta_2). \tag{3}$$

Contact is avoided by stipulating that $h > a_1 + a_2$, where h is defined as the nearest centre-to-centre separation distance between two spheres of radii a_1 and a_2.

Once the magnitude and location of all charges residing on the sphere surfaces are specified then the electrostatic force can be calculated. If sphere surfaces are polarizable then the magnitude and location of charges are determined from Gauss' potential [6] which is discussed next.

1.2. Surface potentials

The electrostatic potential [6] at a point $\vec{x} \equiv (x, y, z)$, due to the spheres, is given by

$$\psi(\vec{x}) = K \int \frac{\mathrm{d}Q_1(\vec{x}_1)}{|\vec{x} - \vec{x}_1|} + K \int \frac{\mathrm{d}Q_2(\vec{x}_2)}{|\vec{x} - \vec{x}_2|} \tag{4}$$

where the potential at $\vec{x} \equiv (x, y, z)$ is the sum of the contributions from the top and bottom spheres, respectively. Equation (4) is valid at all points except when $|\vec{x} - \vec{x}_i| \equiv 0$ ($i = 1, 2$ for the top and bottom spheres, respectively) where the endpoints of the two vectors coincide on the sphere surfaces. Equation (4) can be viewed as the time-independent, instantaneous (action-at-a-distance) and singular propogator solution of the two-particle partial differential equation given by Poisson: $\nabla^2\psi = -4\pi K\rho$, where ∇^2 is the Laplacian and ρ denotes the charge densities, which for our special case, reside only on the surface of the spheres. The charge differential is defined as $\mathrm{d}Q = \mathrm{d}V\rho = \mathrm{d}V(\sigma_1(\vec{x}_1)\delta(|\vec{x}-\vec{x}_1|)+\sigma_2(\vec{x}_2)\delta(|\vec{x}-\vec{x}_2|))$, where $\delta(|\vec{y}|)$ is Dirac's delta function, $\mathrm{d}V$ is a volume element, $\sigma(\vec{y})$ is the charge per unit area and the potential is defined as $\psi = \int_{\text{all space}} \mathrm{d}QG = \int_{\text{all space}} \mathrm{d}V\rho G$, where G is the Green function (the instantaneous propogator or kernel). It follows that $\nabla^2 G = \delta(|\vec{x} - \vec{x}_1|) + \delta(|\vec{x} - \vec{x}_2|)$, for which a particular singular solution is given by $G = K(\frac{1}{|\vec{x}-\vec{x}_1|} + \frac{1}{|\vec{x}-\vec{x}_2|})$, from which ψ follows immediately.

In the case of constant sphere potentials it follows from equation (4) that $4\pi K\sigma_i(\vec{x}_i) = -\frac{\partial\psi}{\partial|\vec{x}|}\big|_{|\vec{x}|=|\vec{x}_i|}$, for $i = 1, 2$. The boundary conditions on the surface of the top and bottom spheres are written [6] as

$$\begin{aligned} V_1 &= K \int \frac{\mathrm{d}Q_1}{R_1} + K \int \frac{\mathrm{d}Q_2}{R_2} \\ V_2 &= K \int \frac{\mathrm{d}Q_1}{R_1'} + K \int \frac{\mathrm{d}Q_2}{R_2'} \end{aligned} \tag{5}$$

where the length quantities R_1, R_1', R_2, and R_2' are shown in figures 1(*b*) and (*c*). Transformation into the spherical coordinate system yields

$$\begin{aligned} V_1 = K \int_0^\pi \int_0^{2\pi} & a_1^2 \sin\theta_1 \,\mathrm{d}\theta_1 \,\mathrm{d}\phi_1 \frac{\sigma_1(\theta_1)}{\sqrt{2a_1^2 - 2a_1^2[\cos\beta\cos\theta_1 + \sin\beta\sin\theta_1\cos(\phi' - \phi_1)]}} \\ & + K \int_0^\pi \int_0^{2\pi} a_2^2 \sin\theta_2 \,\mathrm{d}\theta_2 \,\mathrm{d}\phi_2 \\ & \times \frac{\sigma_2(\theta_2)}{\sqrt{r^2 + a_2^2 - 2a_2 r[\cos\alpha\cos\theta_2 + \sin\alpha\sin\theta_2\cos(\phi - \phi_2)]}} \end{aligned} \tag{6}$$

where $r = \sqrt{a_1^2 + h^2 - 2a_1 h\cos\beta}$, $\cos\alpha = \frac{h - a_1\cos\beta}{\sqrt{a_1^2 + h^2 - 2a_1 h\cos\beta}}$, which must hold for $0 < \beta < \pi$ (figure 1(*c*)), and

$$\begin{aligned} V_2 = K \int_0^\pi \int_0^{2\pi} & a_2^2 \sin\theta_2 \,\mathrm{d}\theta_2 \,\mathrm{d}\phi_2 \frac{\sigma_2(\theta_2)}{\sqrt{2a_2^2 - 2a_2^2[\cos\alpha\cos\theta_2 + \sin\alpha\sin\theta_2\cos(\phi - \phi_2)]}} \\ & + K \int_0^\pi \int_0^{2\pi} a_1^2 \sin\theta_1 \,\mathrm{d}\theta_1 \,\mathrm{d}\phi_1 \\ & \times \frac{\sigma_1(\theta_1)}{\sqrt{r'^2 + a_1^2 - 2a_1 r'[\cos\beta cos\theta_1 + \sin\beta\sin\theta_1\cos(\phi' - \phi_1)]}} \end{aligned} \tag{7}$$

where $r' = \sqrt{a_2^2 + h^2 - 2a_2 h \cos\alpha}$, and $\cos\beta = \frac{h - a_2\cos\alpha}{\sqrt{a_2^2+h^2-2a_2 h\cos\alpha}}$ which must hold for $0 < \alpha < \pi$ (figure 1(*b*)). Equations (6) and (7) couple the potentials, V_1 and V_2, to the charge densities, σ_1 and σ_2, such that for a given surface potential and sphere separation distance the surface charge density is uniquely determined.

Equations (6) and (7) can also be written in terms of Legendre polynomials since V_2 follows from V_1 by obvious interchange of symbols we need only outline the identities used to obtain V_1 given in equation (11) from V_1 given in equation (6). They are

$$\int_0^{2\pi} \frac{\mathrm{d}\phi_1}{\sqrt{2a_1^2 - 2a_1^2[\cos\beta\cos\theta_1 + \sin\beta\sin\theta_1\cos(\phi' - \phi_1)]}} = \sum_{\ell=0}^{\infty} \frac{2\pi}{a_1} P_\ell(\cos\beta) P_\ell(\cos\theta_1) \tag{8}$$

and

$$\begin{aligned}\int_0^{2\pi} & \frac{\mathrm{d}\phi_2}{\sqrt{r^2 + a_2^2 - 2ra_2[\cos\alpha\cos\theta_2 + \sin\alpha\sin\theta_2\cos(\phi - \phi_2)]}} \\ &= 2\pi \sum_{\ell=0}^{\infty} \frac{a_2^\ell}{r^{\ell+1}} P_\ell(\cos\alpha) P_\ell(\cos\theta_2) \\ &= 2\pi \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\ell+m)!}{\ell! m!} \frac{a_2^\ell a_1^m}{h^{\ell+m+1}} P_m(\cos\beta) P_\ell(\cos\theta_2)\end{aligned} \tag{9}$$

where in the last equality we have substituted for the term $\frac{P_\ell(\cos\alpha)}{r^{\ell+1}}$, the identity given by

$$\begin{aligned}\frac{P_\ell(\cos\alpha)}{r^{\ell+1}} &= \frac{P_\ell(\cos\alpha)}{(a_1^2 + h^2 - 2a_1 h\cos\beta)^{\frac{(\ell+1)}{2}}} = \frac{(-1)^\ell}{\ell!}\frac{\partial^\ell}{\partial h^\ell}\frac{1}{\sqrt{a_1^2 + h^2 - 2a_1 h\cos\beta}} \\ &= \frac{(-1)^\ell}{\ell!}\frac{\partial^\ell}{\partial h^\ell}\sum_{m=0}^{\infty} P_m(\cos\beta)\frac{a_1^m}{h^{m+1}} = \sum_{m=0}^{\infty}\frac{(\ell+m)!}{\ell! m!}\frac{a_1^m}{h^{\ell+m+1}} P_m(\cos\beta)\end{aligned} \tag{10}$$

which follows by direct differentiation and substitution for the geometrical relation $h - r\cos\alpha = a_1\cos\beta$. A similar procedure will yield the equation for V_2:

$$V_1 = \sum_{\ell=0}^{\infty} P_\ell(\cos\beta)\left[\frac{A_\ell}{a_1^{\ell+1}} + \sum_{m=0}^{\infty}\frac{(\ell+m)!}{\ell! m!}\frac{a_1^\ell}{h^{\ell+m+1}} B_m\right] \tag{11}$$

$$V_2 = \sum_{\ell=0}^{\infty} P_\ell(\cos\alpha)\left[\frac{B_\ell}{a_2^{\ell+1}} + \sum_{m=0}^{\infty}\frac{(\ell+m)!}{\ell! m!}\frac{a_2^\ell}{h^{\ell+m+1}} A_m\right]. \tag{12}$$

The coefficients in equations (11) and (12) are given by

$$B_m = 2\pi K a_2^{m+2}\int_0^\pi \sin\theta_2\,\mathrm{d}\theta_1\sigma_2(\theta_2)P_m(\cos\theta_2) \tag{13}$$

$$A_m = 2\pi K a_1^{m+2}\int_0^\pi \sin\theta_1\,\mathrm{d}\theta_1\sigma_1(\theta_1)P_m(\cos\theta_1). \tag{14}$$

Orthonormality relations of Legendre polynomials, $\int_{-1}^{1}\mathrm{d}\cos\theta\, P_m(\cos\theta)P_k(\cos\theta) = \frac{2}{2m+1}\delta_{m,k}$, applied to equations (11) and (12) implies that

$$V_1\delta_{\ell,0} = \frac{A_\ell}{a_1^{\ell+1}} + \sum_{m=0}^{\infty} B_m\frac{(\ell+m)!}{\ell! m!}\frac{a_1^\ell}{h^{\ell+m+1}} \tag{15}$$

$$V_2\delta_{\ell,0} = \frac{B_\ell}{a_2^{\ell+1}} + \sum_{m=0}^{\infty} A_m\frac{(\ell+m)!}{\ell! m!}\frac{a_2^\ell}{h^{\ell+m+1}}. \tag{16}$$

An expression for the charge densities in equations (14) and (13) are obtained by using the distributive properties of Legendre polynomials, $\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(\cos\alpha) P_\ell(\cos\theta_2) = \delta(\cos\alpha - \cos\theta_2)$, and yields

$$\sigma_1(\theta_1) = \frac{1}{4\pi K} \sum_{m=0}^{\infty} \frac{(2m+1)A_m}{a_1^{m+2}} P_m(\cos\theta_1) \tag{17}$$

$$\sigma_2(\theta_2) = \frac{1}{4\pi K} \sum_{m=0}^{\infty} \frac{(2m+1)B_m}{a_2^{m+2}} P_m(\cos\theta_2). \tag{18}$$

We can use Gauss' successive approximations to solve for charge densities as follows. From equations (15) and (16) we have

$$A_\ell = V_1 a_1 \delta_{\ell,0} - V_2 a_2 \frac{a_1^{2\ell+1}}{h^{\ell+1}} + \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} A_k \frac{(\ell+m)!}{\ell! m!} \frac{(k+m)!}{k! m!} \frac{a_1^{2\ell+1}}{h^{\ell+k+1}} \frac{a_2^{2m+1}}{h^{2m+1}} \tag{19}$$

$$B_\ell = V_2 a_2 \delta_{\ell,0} - V_1 a_1 \frac{a_2^{2\ell+1}}{h^{\ell+1}} + \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} B_k \frac{(\ell+m)!}{\ell! m!} \frac{(k+m)!}{k! m!} \frac{a_2^{2\ell+1}}{h^{\ell+k+1}} \frac{a_1^{2m+1}}{h^{2m+1}}. \tag{20}$$

Equations (17) and (19) give

$$\sigma_1(\theta_1) = \frac{V_1}{4\pi K a_1} - \frac{V_2}{4\pi K a_1} \frac{a_2}{h} \sum_{\ell=0}^{\infty} (2\ell+1) \frac{a_1^\ell}{h^\ell} P_\ell(\cos\theta_1)$$
$$+ \frac{1}{4\pi K} \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{a_1^{\ell+2}} P_\ell(\cos\theta_1) \left[\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} A_k \frac{(\ell+m)!}{\ell! m!} \frac{(k+m)!}{k! m!} \frac{a_1^{2\ell+1}}{h^{\ell+k+1}} \frac{a_2^{2m+1}}{h^{2m+1}} \right].$$

Substituting from equations (14) and (13) for A_k, we obtain

$$\sigma_1(\cos\theta_1) = \frac{V_1}{4\pi K a_1} - \frac{V_2}{4\pi K a_1} \frac{a_2}{h} \sum_{\ell=0}^{\infty} (2\ell+1) \frac{a_1^\ell}{h^\ell} P_\ell(\cos\theta_1)$$
$$+ \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta_1) \left[\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\ell+m)!}{\ell! m!} \frac{(k+m)!}{k! m!} \frac{a_1^{\ell+k+1}}{h^{\ell+k+1}} \frac{a_2^{2m+1}}{h^{2m+1}} \right]$$
$$\times \int_0^\pi \mathrm{d}\theta_2 \sin\theta_2 \sigma_1(\theta_2) P_k(\cos\theta_2).$$

Letting $\alpha_1 = \frac{a_1}{h}$, $\alpha_2 = \frac{a_2}{h}$, $x_1 = \cos\theta_1$ and $x_2 = \cos\theta_2$, we have

$$\sigma_1(x_1) = \frac{V_1}{4\pi K a_1} - \frac{V_2}{4\pi K a_1} \alpha_2 \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_1^\ell P_\ell(x_1)$$
$$+ \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x_1) \left[\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\ell+m)!}{\ell! m!} \frac{(k+m)!}{k! m!} \alpha_1^{\ell+k+1} \alpha_2^{2m+1} \right]$$
$$\times \int_{-1}^{1} \mathrm{d}x_2 \sigma_1(x_2) P_k(x_2). \tag{21}$$

Equation (21) can be viewed as an inhomogeneous integral equation of Fredholm type with singular kernel

$$\sigma_1(x_1) = \Sigma_0'(x) + \lambda \int_{-1}^{1} \mathrm{d}x_2 K(x_1, x_2) \sigma_1(x_2) \tag{22}$$

where

$$\Sigma_0'(x_1) = \frac{V_1}{4\pi K a_1} - \frac{V_2}{4\pi K a_1} \alpha_2 \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_1^\ell P_\ell(x_1)$$

$$\lambda K(x_1, x_2) = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(x_1)\left[\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\ell+m)!}{\ell!m!}\frac{(k+m)!}{k!m!}\alpha_1^{\ell+k+1}\alpha_2^{2m+1}\right]P_k(x_2).$$

Because the iterative solution for the Fredholm integral equation is known [7] the $(N+1)$th-order approximation of equation (22) can be written as

$$\Sigma_{N+1}(x_1) = \Sigma'_0(x_1) + \Sigma'_1(x_1) + \Sigma'_2(x_1) + \cdots + \Sigma'_{N+1}(x_1) \tag{23}$$

where

$$\Sigma'_1(x_1) = \lambda \int_{-1}^{1} \mathrm{d}x_2 K(x_1, x_2)\Sigma'_0(x_2)$$

$$\Sigma'_2(x_1) = \lambda \int_{-1}^{1} \mathrm{d}x_2 K(x_1, x_2)\Sigma'_1(x_2)$$

$$\vdots$$

$$\Sigma'_{N+1}(x_1) = \lambda \int_{-1}^{1} \mathrm{d}x_2 K(x_1, x_2)\Sigma'_N(x_2)$$

and

$$\sigma_1(x_1) = \Sigma_{N+1}(x_1) + \lambda^N \int_{-1}^{1}\int_{-1}^{1}\ldots\int_{-1}^{1} \mathrm{d}x_2\,\mathrm{d}x_3\ldots\mathrm{d}x_{N+1}\,K(x_1,x_2)K(x_2,x_3)\ldots K(x_N,x_{N+1})\sigma_1(x_{N+1}). \tag{24}$$

By substitution the charge densities, $\sigma_1(x_1)$ and $\sigma_2(x_1)$, follow immediately:

$$\sigma_1(x_1) = \frac{V_1}{4\pi K a_1}\left[1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} z'_k z_k \frac{1 - z'^2_m}{[1 + z'^2_m - 2z'_m x_1]^{\frac{3}{2}}}\right] - \frac{V_2}{4\pi K a_1}\frac{\alpha_2}{u'_0 u_1}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{m} u'_{k-1}u_k \frac{1 - u^2_m}{[1 + u^2_m - 2u_m x_1]^{\frac{3}{2}}}\right] \tag{25}$$

and

$$\sigma_2(x_1) = \frac{V_2}{4\pi K a_2}\left[1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} u'_k u_k \frac{1 - u'^2_m}{[1 + u'^2_m - 2u'_m x_1]^{\frac{3}{2}}}\right] - \frac{V_1}{4\pi K a_2}\frac{\alpha_1}{z'_0 z_1}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{m} z'_{k-1}z_k \frac{1 - z^2_m}{[1 + z^2_m - 2z_m x_1]^{\frac{3}{2}}}\right] \tag{26}$$

where $z'_0 = u'_0 = 1$, $z_1 = \alpha_2$ and $z_N = \frac{\alpha_2}{1-\alpha_1 z'_{N-1}}$; $z'_1 = \frac{\alpha_1}{1-\alpha_2 z_1}$ and $z'_N = \frac{\alpha_1}{1-\alpha_2 z_N}$; $u_1 = \alpha_1$ and $u_N = \frac{\alpha_1}{1-\alpha_2 u'_{N-1}}$; $u'_1 = \frac{\alpha_2}{1-\alpha_1 u_1}$ and $u'_N = \frac{\alpha_2}{1-\alpha_1 u_N}$ for $N \geqslant 2$. The net charges Q_1 and Q_2 follow by integration of equations (25) and (26) ($Q_i = 2\pi a_i^2 \int_{-1}^{1} \mathrm{d}x\,\sigma_i(x)$; $i = 1, 2$):

$$Q_1 = \frac{V_1 a_1}{K}\left[1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} z_k z'_k\right] - \frac{V_2 a_1}{K}\frac{\alpha_2}{u'_0 u_1}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{m} u'_{k-1}u_k\right] \tag{27}$$

$$Q_2 = \frac{V_2 a_2}{K}\left[1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} u_k u'_k\right] - \frac{V_1 a_2}{K}\frac{\alpha_1}{z'_0 z_1}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{m} z'_{k-1}z_k\right]. \tag{28}$$

For equivalent sphere potentials and radii the net charge reads

$$Q_1 = Q_2 = \frac{V_1 a_1}{K}\left[1 + \sum_{m=1}^{\infty}(-1)^m \frac{1}{m+1}\right] = \frac{V_1 a_1}{K}\ln 2 \tag{29}$$

which is in exact agreement with Kelvin's result [10].

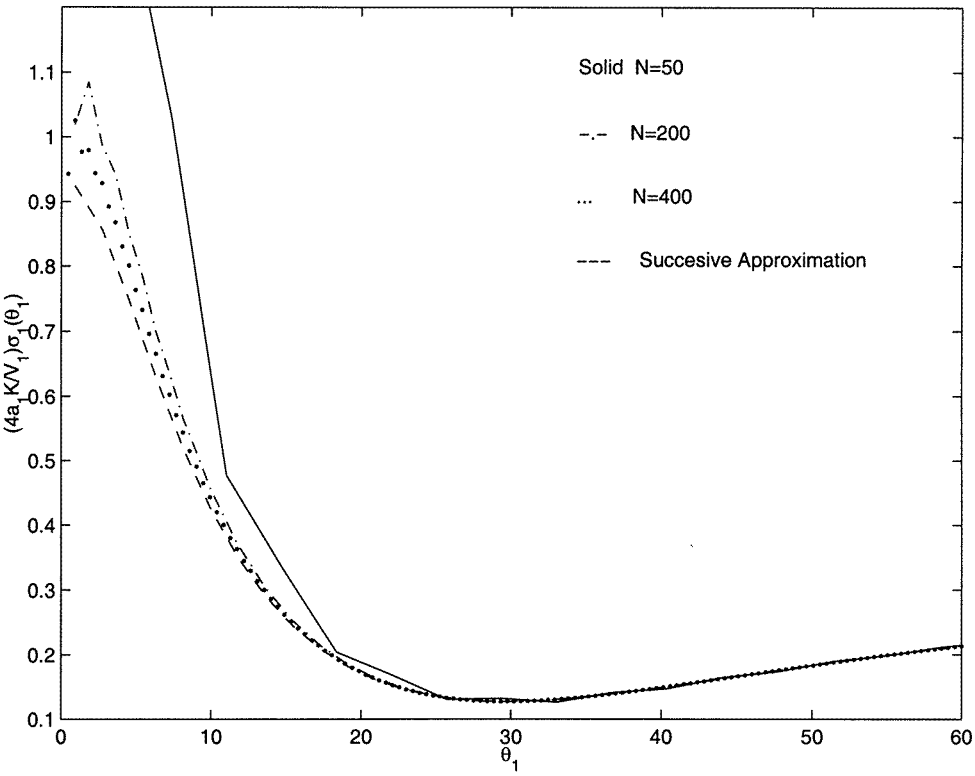

Figure 2. Evaluation of the charge density distribution for two conducting spheres of equal size, $a_1 \equiv a_2$, and ratio of sphere potentials, $V_2/V_1 = 0.927$. The evaluation is carried out using the surface integral method (floating point average) for $N = 50$ (——), $N = 200$ (— · —) and $N = 400$ (· · · · · ·) and using the Fredholm-type expression for $N = 100$ (— — —).

1.3. Efficiency

The efficiency of the Fredholm-type expression for the charge density distribution was compared with the surface integral method, or moment method [8], by considering the charge density distribution for the case where both spheres are of equal size, $a_1 = a_2 = a$, held at constant but unequal potential, $V_2/V_1 = 5100/5500$, at a sphere–sphere separation distance of $d/a = 0.025$. Calculations for the charge density distribution on the sphere with the higher potential using the surface integral method were carried out for several values of N, where N is the number of subdivisions of the polar angle, θ, where $0 < \theta < \pi$, and step-size Δ is given by $\Delta = \frac{\pi}{N-1}$. The surface integral requires filling an $N \times N$ matrix followed by an inversion which requires approximately $N^3 + \frac{N(N+1)}{2}$ operations using forward elimination and backward substitution in Gauss' procedure [9]. In figure 2 the calculated charge density distribution for $N = 50$, 200 and 400 are shown as a function of polar angle. It is immediately apparent that even smaller step sizes (larger N) are required for the surface integral method to approach the asymptotically correct value for the charge density distribution obtained from the solution of the Fredholm integral equation using only 100 intervals. Using the Fredholm-type expression requires only $\frac{N(N+1)}{2}$ operations for the same order accuracy which means that the total number of operations has been reduced from approximately N^3, to approximately N^2.

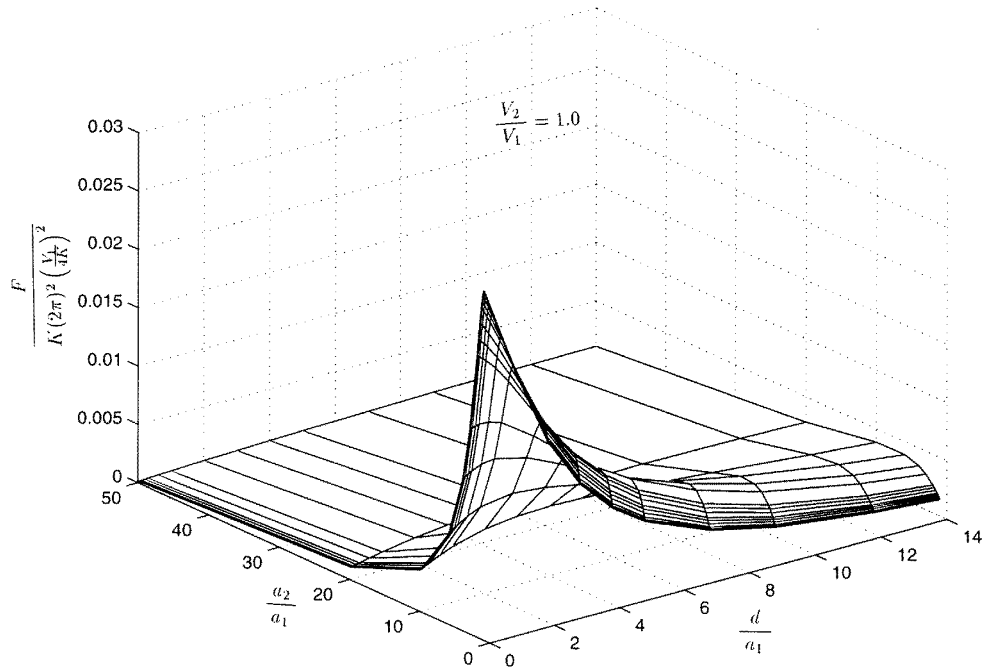

Figure 3. The Coulomb force between two dissimilar, fully polarizable, spheres as a function of the surface potential, V_2/V_1, and separation distance, d/a_1, where d is the surface-to-surface separation, and a_1 is the radius of the top sphere. Force is scaled with a factor $f = K(2\pi)^2(V_1/4K)^2$ which corresponds to $f = 2.74 \times 10^{-10}$ for a particle with radius 10×10^{-6} m and potential $V_1 = 1$ mV. K is the factor $1/4\pi\epsilon_0$ approximately equal to 9×10^9 Vm C^{-1}. The force is plotted for $V_1/V_2 = 1.0$.

2. Coulomb force

The electrostatic force is evaluated in a second calculation where we integrate the effect of the known charge distribution using the expression for the Coulomb force. Direct substitution of $\sigma_1(x_1)$ and $\sigma_2(x_1)$ from equations (25) and (26) into equation (3), yields, after integration over θ

$$F = \frac{V_1 V_2}{K}\alpha_1\alpha_2\Bigg[1 + \sum_{m=1}^{\infty}\frac{\prod_{k=1}^{m}u'_k u_k}{(1-\alpha_2 u'_m)^2} + \sum_{m=1}^{\infty}\frac{\prod_{k=1}^{m}z'_k z_k}{(1-\alpha_1 z'_m)^2} + \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\prod_{k=1}^{m}z'_k z_k\prod_{i=1}^{\ell}u'_i u_i}{(1-\alpha_1 z'_m-\alpha_2 u'_\ell)^2}$$
$$+\frac{\alpha_1\alpha_2}{u'_0 u_1 z'_0 z_1}\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\prod_{k=1}^{m}u'_{k-1}u_k\prod_{i=1}^{\ell}z'_{i-1}z_i}{(1-\alpha_1 u_m-\alpha_2 z_\ell)^2}\Bigg]$$
$$-\frac{V_1 V_1\alpha_1}{K z'_0 z_1}\alpha_1\alpha_2\Bigg[\sum_{m=1}^{\infty}\frac{\prod_{k=1}^{m}z'_{k-1}z_k}{(1-\alpha_2 z_m)^2} + \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\prod_{k=1}^{m}z'_k z_k\prod_{i=1}^{\ell}z'_{i-1}z_i}{(1-\alpha_1 z'_m-\alpha_2 z_\ell)^2}\Bigg]$$
$$-\frac{V_2 V_2\alpha_2}{K u'_0 u_1}\alpha_1\alpha_2\Bigg[\sum_{m=1}^{\infty}\frac{\prod_{k=1}^{m}u'_{k-1}u_k}{(1-\alpha_1 u_m)^2} + \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\prod_{k=1}^{m}u'_k u_k\prod_{i=1}^{\ell}u'_{i-1}u_i}{(1-\alpha_2 u'_m-\alpha_1 u_\ell)^2}\Bigg]. \qquad (30)$$

Equation (30) is valid for the electrostatic force between two polarizable spheres held at constant potentials, V_1 and V_2. Experimentally, the constant potential limit describes the physical situation of two polarizable spheres connected to a voltage power supply. In figure 3 the electrostatic force is plotted as a function of relative sphere size, a_2/a_1, and separation

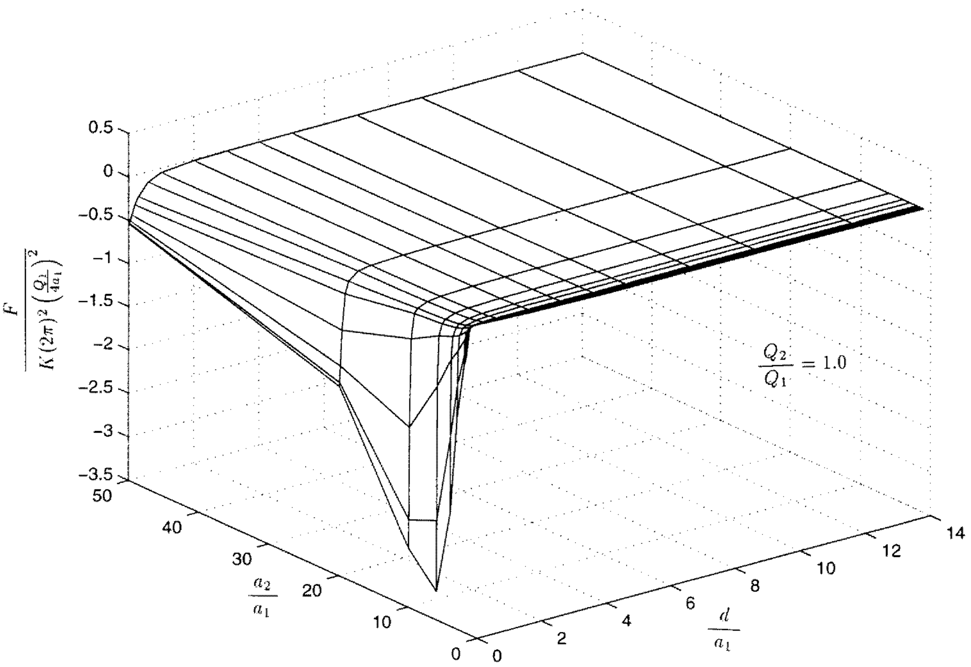

Figure 4. The Coulomb force between two dissimilar, fully polarizable, spheres as a function of the effective charge density, Q_2/Q_1, and particle size ratio, a_2/a_1, where a_1 and a_2 are the radii of the top and bottom spheres, respectively. Force is scaled with a factor $f = K(2\pi)^2(Q_1/a_1)^2$, which corresponds to $f = 9.11 \times 10^{-17}$ N for a sphere with radius 10×10^{-6} m and net charge 1.6×10^{-19} C. K is the factor $1/4\pi\epsilon_0$ approximately equal to 9×10^9 Vm C^{-1}.

distance, d/a_1. The most important feature is that the Coulomb force is everywhere repulsive and monotonic with respect to separation distance but not with respect to relative size ratio. Furthermore, in the limit of sphere–plate interaction, i.e. when $a_2/a_1 = \infty$, the force at near contact approaches zero.

Alternatively, the polarizable spheres may be held at constant charge, Q_1, and Q_2. Experimentally, the constant charge limit describes the charged state of conductive particles such as aerosols [11] or water in oil emulsions. If particle charges are specified equation (30) will still hold after the necessary substitutions are made from equations (27) and (28):

$$V_1 = \frac{1}{D}\left[\frac{KQ_1}{a_1}\left(1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} u_k u_k'\right) + \frac{KQ_2}{a_2}\frac{\alpha_2}{u_0' u_1}\sum_{m=1}^{\infty}\prod_{k=1}^{m} u_{k-1}' u_k\right] \tag{31}$$

$$V_2 = \frac{1}{D}\left[\frac{KQ_2}{a_2}\left(1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} z_k z_k'\right) + \frac{KQ_1}{a_1}\frac{\alpha_1}{z_0' z_1}\sum_{m=1}^{\infty}\prod_{k=1}^{m} z_{k-1}' z_k\right] \tag{32}$$

where

$$D = \left[1 + \sum_{m=1}^{\infty}\prod_{k=1}^{m} z_k z_k'\right]\left[1 + \sum_{\ell=1}^{\infty}\prod_{i=1}^{\ell} u_i u_i'\right] - \frac{\alpha_1\alpha_2}{z_0' z_1 u_0' u_1}\sum_{m=1}^{\infty}\prod_{k=1}^{m} u_{k-1}' u_k \sum_{\ell=1}^{\infty}\prod_{i=1}^{\ell} z_{i-1}' z_i$$

assuming no charge loss, $\Delta Q = 0$, during the interaction. In figure 4 we plot the electrostatic force between spheres having equal charge, $Q_2/Q_1 = 1$ with respect to relative sphere size, a_2/a_1, and separation distance, d/a_1. Noticeably, it is only for equivalent sized spheres, $a_1 \equiv a_2$, that the pair interaction is everywhere repulsive to monotonically approach zero at

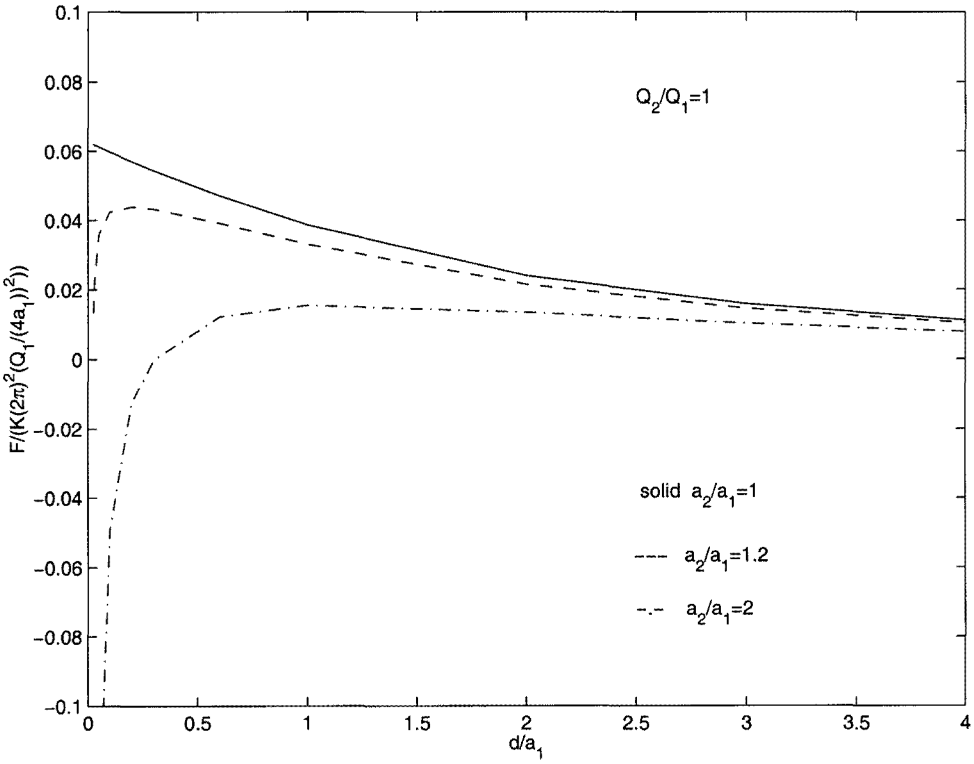

Figure 5. The Coulomb force between two, fully polarizable, spheres with effective charge density, $Q_2/Q_1 = 1.0$, as a function of particle size ratio, a_2/a_1, and separation distance, d/a_1. The force is plotted for particle size ratios: $(a)\ \frac{a_2}{a_1} = 1$ (——), $(b)\ \frac{a_2}{a_1} = 1.2$ (— — —) and $(c)\ \frac{a_2}{a_1} = 2$ (— · —). See also figure 4.

large separations. Any difference in size, $a_1 \neq a_2$, will give rise to a near-field attraction. Because most particulate systems are polydisperse in nature the electrostatic contribution to the pair interaction is, therefore, always expected to be attractive at close separations. For clarity, in figure 5 we plot a few special cases of force that appear in figure 4, for: $(a)\ \frac{a_2}{a_1} = 1$, $(b)\ \frac{a_2}{a_1} = 1.2$ and $(c)\ \frac{a_2}{a_1} = 2$.

For the sake of completeness we also give the limiting case of the expression for Coulomb force, equation (30), for two equivalent spheres in 'contact'. The summed terms in equation (30) are arranged in their proper order [10] to obtain the contact force which reads

$$
\begin{aligned}
F &= \frac{V^2}{K}\left[\frac{1}{4} + 2\sum_{m=1}^{\infty}(-1)^m \frac{(m+1)}{(m+2)^2} + \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{\ell+m}\frac{(\ell+1)(m+1)}{(\ell+m+2)^2}\right] \\
&= \frac{V^2}{K}\left[\frac{1}{4} + 2\left(\frac{\pi^2}{12} - \ln 2 - \frac{1}{4}\right) + \frac{1}{6}\left(\frac{5}{4} + 13\ln 2 - \pi^2\right)\right] \\
&= \frac{V^2}{6K}\left(\ln 2 - \frac{1}{4}\right) \qquad (33)
\end{aligned}
$$

which is in exact agreement with Kelvin's result [10].

3. Conclusion

The fundamental problem of the electrostatic force between two polarizable spheres is solved by casting the expression for the charge density distribution in the form of the Fredholm integral equation of the second kind. The result is an expression of the Coulomb force in a series form which converges rapidly. From a computational standpoint it is most important that the Fredholm-type expression can be evaluated two to four orders of magnitude faster than the surface integral method depending upon the desired accuracy at small polar angles. Such savings in computational speed may be significant in models of particle aggregation and suspension stability where pair-interaction forces are evaluated to predict phase transitions.

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